Breathing Vibrations of Pressurized Partially Filled Tanks

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The differential equations governing the vibrations of an axially symmetrical shell filled with one or more nonmixable, nonviscous, compressible fluids under pressure are established within the scope of the linear small displacement theory. The simply supported, circular cylindrical shell containing either one or two fluids under pressure is then analyzed, and the characteristic equations are established, taking into consideration the three inertia as well as the membrane and bending terms. As was to be expected, the coefficients of the characteristic equation which, for the empty shell are constants, now become Bessel functions of the frequency representing the interaction between fluid and shell. For the case of two fluids, in addition to two characteristic equations, a continuity condition must also be satisfied. The solution of the completely filled tank is obtained in a straightforward manner. For the case of two fluids, the solutions are obtained by first determining the solutions of the characteristic equations and then substituting those values into the continuity condition. It is apparent from the system of equations that, in general, there will not be a continuous spectrum of frequencies as function of fluid level for any particular mode shape. An iteration procedure is outlined and applied to a particular example.

Introduction

THE problem of vibrations of a tank containing one or more fluids under pressure is of fundamental interest in a variety of problems of present day technology. A prime example of this is the design of large fuel tanks for liquid propellant rockets, where the requirement of a light weight structure and the large percentage of the total mass contributed by the propellant call for an analysis of the dynamic interaction between fluid and elastic container which is as accurate as possible.

The breathing vibrations of a pressurized cylindrical shell containing a heavy liquid have been investigated by Berry and Reissner[1] using shallow shell theory. Lindholm et al.[2] carried on an experimental investigation of partially filled cylindrical tanks. Chu[3] analyzed a cylindrical shell partially filled with an incompressible fluid, using Donnell's[4, 8] equations as extended by Yu[5] to the dynamic case and, following Reissner,[6] neglected the axial and circumferential inertial terms.

In the present paper, the differential equations of motion are first established, within the scope of the linear small displacement theory, for an axisymmetric pressurized shell filled with one or more nonmixable, nonviscous, compressible fluids. Thereafter, the specific case is considered of a closed, simply supported, circular cylindrical shell 1) completely filled with a single fluid and 2) containing two nonmixable fluids.

Equations of Motion

A. Shell

Within the scope of the linear elastic theory the differential equations of motion of a thin shell can be written[7, 8]:

$$\sum_{j=1}^{3}\left[L_{ij} - \delta_{ij}m_s\frac{\partial^2}{\partial t^2}\right]u_j = p_i \qquad i = 1, 2, 3 \tag{1}$$

where

u_j = displacement component in the j direction
p_i = component of the external force in the i direction
m_s = inertial mass per unit area of the shell
L_{ij} = linear differential operators in the shell coordinates associated with the shell geometry
t = time
δ_{ij} = (Kronecker's delta), value 1 for $i = j$ and zero otherwise

The natural modes are obtained as the solution of the set of three homogeneous differential equations:

$$\sum_{j=1}^{3}[L_{ij} + \delta_{ij}m_s\omega_k^2]U_{jk} = 0 \tag{2}$$

obtained by substituting

$$u_{jk} = U_{jk}\sin\omega_k t \tag{3}$$

into Eq. (1).

The functions U_{jk}, satisfying the homogeneous system (2) and associated boundary conditions, characterize the mode shape associated with the frequency ω_k.

The requirement that Eqs. (2) have nontrivial solutions leads to the frequency equation associated with the problem.

B. Fluid

The small motion of a compressible nonviscous fluid is governed by the equation[9]

$$[(\partial^2/c_0^2\partial t^2) - \nabla^2]q = 0 \tag{4}$$

where

q = $p-p_0$ = pressure fluctuation about the mean value p_0
c_0 = velocity of sound in the fluid at the pressure p_0
∇^2 = Laplacian operator

The acceleration of a fluid particle in the x_i direction is given by

$$\partial^2 u_i/\partial t^2 = -\partial q/\rho_0\partial x_i \tag{5}$$

where ρ_0 is the density of the fluid, and u_i is the component of displacement in the x_i direction.

C. Interaction between Fluid and Shell

The interaction between the fluid and the shell is obtained by equating the normal accelerations and pressures of fluid and shell at the shell surface. Thus,

$$\frac{\partial^2}{\partial t^2}u_{\eta^s} = \frac{\partial^2}{\partial t^2}u_{\eta^f} = -\frac{1}{\rho_0}\frac{\partial q}{\partial \eta} \tag{6}$$

$$p_{\eta^s} = p_{\eta^f} \tag{7}$$

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at the shell surface. The indices s and f refer to shell and fluid, and η denotes the direction of the shell outer normal.

If there are two nonmixable fluids inside the shell, then, under the assumption that the dynamic overpressure as well as the pressure due to gravity are small as compared with the static pressure p_0, the following condition must be satisfied at the interface:

$$u_\eta^{f_1} = u_\eta^{f_2} \tag{8}$$

which, by Eq. (5), implies

$$\partial q_1/\rho_1 \partial \eta_1 = \partial q_2/\rho_2 \, \partial \eta_1 \tag{9}$$

where η_1 is the direction normal to the interface.

Completely Filled Cylindrical Tank

For a cylindrical shell, the differential equations (1) can be written as[7, 8]

$$\left.\begin{aligned}
&\left[\frac{\partial^2}{\partial x^2} + \frac{1-\nu}{2a^2}\frac{\partial^2}{\partial \theta^2} - \frac{m_s}{D}\frac{\partial^2}{\partial t^2}\right] u + \frac{1+\nu}{2a}\frac{\partial^2}{\partial x \partial \theta} v - \frac{\nu}{a}\frac{\partial}{\partial x} w = 0 \\
&\frac{1+\nu}{2a}\frac{\partial^2}{\partial x\,\partial \theta} u + \left[\frac{1-\nu}{2}\frac{\partial^2}{\partial x^2} + \frac{1}{a^2}\frac{\partial^2}{\partial \theta^2} - \frac{m_s}{D}\frac{\partial^2}{\partial t^2}\right] v - \frac{1}{a^2}\frac{\partial}{\partial \theta} w = 0 \\
&\frac{\nu}{a}\frac{\partial}{\partial x} u + \frac{1}{a^2}\frac{\partial}{\partial \theta} v - \left[\frac{1}{a^2} + \frac{h^2}{12}\nabla^4 - \frac{p_0 a}{2D}\left(\frac{\partial^2}{\partial x^2} + \frac{2}{a^2}\frac{\partial^2}{\partial \theta^2}\right) + \frac{m_s}{D}\frac{\partial^2}{\partial t^2}\right] w = \frac{q}{D}
\end{aligned}\right\} \tag{10}$$

where

$u, v, w,$ = displacement components in the x, θ, and negative r directions (see Fig. 1)
$a, h, m_s,$ = radius, wall thickness, and mass per unit area of shell
ν, D = $Eh/(1 - \nu^2)$ = Poisson's ratio, and membrane stiffness, respectively
p_0 = static pressure
q = dynamic overpressure acting on the shell

$$\nabla^4 = \frac{\partial^4}{\partial x^4} + \frac{2}{a^2}\frac{\partial^4}{\partial x^2\,\partial \theta^2} + \frac{1}{a^4}\frac{\partial^4}{\partial \theta^4}$$

For the simply supported shell, the solutions of the associated homogeneous equations are of the form

$$\begin{aligned}
u_{mn} &= U_{mn}\cos(m\pi x/l)\cos n\theta \sin\omega_{mn}t \\
v_{mn} &= V_{mn}\sin(m\pi x/l)\sin n\theta \sin\omega_{mn}t \\
w_{mn} &= W_{mn}\sin(m\pi x/l)\cos n\theta \sin\omega_{mn}t
\end{aligned} \tag{11}$$

where l is the length of the cylindrical shell, and U_{mn}, V_{mn}, and W_{mn} are constant coefficients. The fluid equation (4) in cylindrical coordinates has a solution of the form

$$q_{mn} = Q_{mn}J_n\,[\lambda(r/a)]\sin(m\pi x/l)\cos n\theta \sin\omega_{mn}t \tag{12}$$

where the J_n is the Bessel function of the first kind and order n, and

$$\lambda^2 = \lambda_{mn}{}^2 = \left(\frac{\omega_{mn}a}{c_0}\right)^2 - \left(\frac{m\pi a}{l}\right)^2 = \left(\frac{c_s}{c_0}\right)^2\Omega^2 - \beta^2 \tag{13}$$

with

$$\beta = \frac{m\pi a}{l} \qquad \Omega = \frac{\omega_{mn}a}{c_s} \qquad c_s{}^2 = \frac{D}{m_s} \tag{14}$$

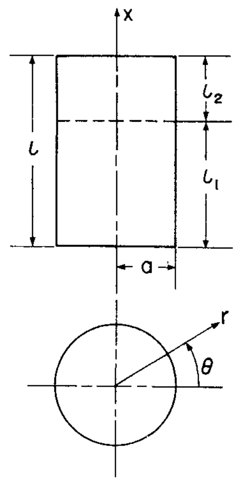

Fig. 1 Coordinate system.

Substituting (11) and (12) into (6) and (7) we obtain

$$q_{mn} = -\omega_{mn}{}^2 a\rho_0 B(\lambda) W_{mn}\sin(m\pi x/l)\cos n\theta \sin\omega_{mn}t \tag{15}$$

$$B(\lambda) = B_b = J_n(\lambda)/\lambda J_n{}'(\lambda) \tag{16}$$

where the prime indicates differentiation with respect to the argument. Substituting Eqs. (15) and (11) into (10), we obtain

$$\left.\begin{aligned}
(\Omega^2 - a_{11})U_{mn} + a_{12}V_{mn} + a_{13}W_{mn} &= 0 \\
a_{21}U_{mn} + (\Omega^2 - a_{22})V_{mn} + a_{23}W_{mn} &= 0 \\
a_{31}U_{mn} + a_{32}V_{mn} + ([1 + \mu B]\Omega^2 - a_{33})W_{mn} &= 0
\end{aligned}\right\} \tag{17}$$

where

$$a_{11} = \beta^2 + \frac{1-\nu}{2}n^2 \qquad a_{12} = a_{21} = \frac{1+\nu}{2}\beta n$$

$$a_{22} = \frac{1-\nu}{2}\beta^2 + n^2 \qquad a_{13} = a_{31} = -\nu\beta$$

$$a_{33} = 1 + \frac{p_0 a}{2D}(2n^2 + \beta^2) + \frac{h^2}{12a^2}(n^2 + \beta^2)^2$$

$$a_{23} = a_{32} = n \qquad \mu = a\rho_0/m_s$$

The requirement that (17) has a nontrivial solution leads to the characteristic equation

$$[(1 + \mu B)\Omega^2 - a_{33}][\Omega^2 - (n^2 + \beta^2)] \times \left[\Omega^2 - \frac{1-\nu}{2}(n^2 + \beta^2)\right] - (n^2 + \nu^2\beta^2)\Omega^2 + \frac{1-\nu}{2}(n^4 + 2n^2\beta^2 + \nu^2\beta^4) = 0 \tag{18}$$

where

$$\left.\begin{aligned}
a_{33} &= 1 + \varkappa(2n^2 + \beta^2) + \epsilon(n^2 + \beta^2)^2 \\
\varkappa &= \frac{p_0 a}{2D} \qquad \epsilon = \frac{h^2}{12a^2}
\end{aligned}\right\} \tag{19}$$

For the axisymmetric case ($n = 0$) Eq. (18) reduces to

$$[(1 + \mu B)\Omega^2 - a_{33}](\Omega^2 - \beta^2) - \nu^2\beta^2 = 0 \tag{18'}$$

with

$$a_{33} = 1 + \varkappa\beta^2 + \epsilon\beta^4 \tag{19'}$$

The solution of the problem requires that Eqs. (16) and (18) be simultaneously satisfied.

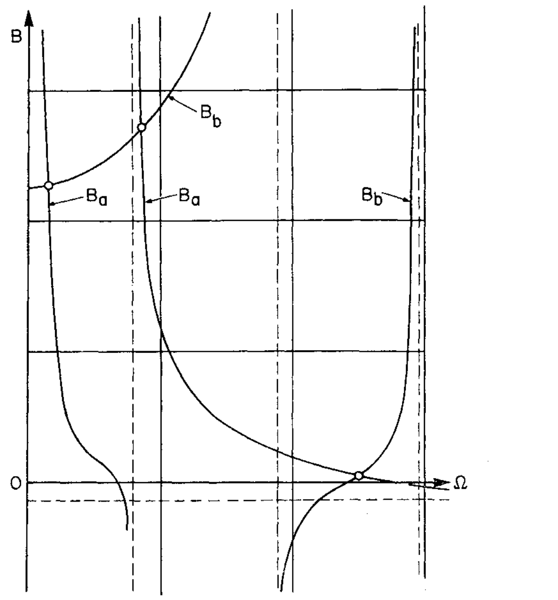

Fig. 2 Full shell: sketch of solution for axisymmetric case.

Partially Filled Cylindrical Tank

If we let the index $i = 1$ and 2 indicate the bottom and top parts of the cylindrical shell as shown in Fig. 1; the differential equations (1) can then be written as

$$\left[\frac{\partial^2}{\partial x^2} + \frac{1-\nu}{2a^2}\frac{\partial^2}{\partial\theta^2} - \frac{m_s}{D}\frac{\partial^2}{\partial t^2}\right]u_i + \frac{1+\nu}{2a}\frac{\partial^2}{\partial x\partial\theta}v_i - \frac{\nu}{a}\frac{\partial}{\partial x}w_i = 0$$

$$\frac{1+\nu}{2a}\frac{\partial^2}{\partial x\,\partial\theta}u_i + \left[\frac{1-\nu}{2}\frac{\partial^2}{\partial x^2} + \frac{1}{a^2}\frac{\partial^2}{\partial\theta^2} - \frac{m_s}{D}\frac{\partial^2}{\partial t^2}\right]v_i - \frac{1}{a^2}\frac{\partial}{\partial\theta}w_i = 0 \qquad (20)$$

$$\frac{\nu}{a}\frac{\partial}{\partial x}u_i + \frac{1}{a^2}\frac{\partial}{\partial\theta}v_i - \left[\frac{1}{a^2} + \frac{h^2}{12}\nabla^4 - \frac{p_0a}{2D}\left(\frac{\partial^2}{\partial x^2} + \frac{2}{a^2}\frac{\partial^2}{\partial\theta^2}\right) + \frac{m_s}{D}\frac{\partial^2}{\partial t^2}\right]w_i = \frac{q_i}{D}$$

where the meaning of the symbols is the same as for the completely filled tank.

For the simply supported shell the solutions of the associated homogeneous equations can be assumed in the form of

$$\left.\begin{aligned} u_1 &= U_1\cos\beta_1(x/a)\cos n\theta\sin\omega t \\ v_1 &= V_1\sin\beta_1(x/a)\sin n\theta\sin\omega t \\ w_1 &= W_1\sin\beta_1(x/a)\cos n\theta\sin\omega t \end{aligned}\right\} \qquad (21a)$$

$$\left.\begin{aligned} u_2 &= U_2\cos[\beta_2(l-x)/a]\cos n\theta\sin\omega t \\ v_2 &= V_2\sin[\beta_2(l-x)/a]\sin n\theta\sin\omega t \\ w_2 &= W_2\sin[\beta_2(l-x)/a]\cos n\theta\sin\omega t \end{aligned}\right\} \qquad (21b)$$

At $x = l_1$ we require only that

$$w_1 = w_2 \qquad (\partial/\partial x)w_1 = (\partial/\partial x)w_2 \qquad (22)$$

Equation (22) yields

$$\beta_1\cot\beta_1(l_1/a) + \beta_2\cot\beta_2(l_2/a) = 0 \qquad (23)$$

As before, we assume solutions of the fluid equation (4) of the form

$$q_1 = Q_1J_n(\lambda_1r/a)\sin\beta_1(x/a)\cos n\theta\sin\omega t \qquad (24a)$$

$$q_2 = Q_2J_n(\lambda_2r/a)\sin[\beta_2(l-x)/a]\cos n\theta\sin\omega t \qquad (24b)$$

where

$$\lambda_1^2 = (\omega a/c_1)^2 - \beta_1^2 = (c_s/c_1)^2\Omega^2 - \beta_1^2 \qquad (25a)$$

$$\lambda_2^2 = (\omega a/c_2)^2 - \beta_2^2 = (c_s/c_2)^2\Omega^2 - \beta_2^2 \qquad (25b)$$

$$\Omega^2 = (\omega a/c_s)^2 \qquad (25c)$$

where c_1 and c_2 are the velocities of sound of the respective fluids. The satisfaction of Eq. (9) for all values of r requires that

$$\lambda_1 = \lambda_2 = \lambda \qquad (26)$$

Substituting (21) and (24) into (6) and (7) we obtain

$$q_1 = -\omega^2a\rho_1B(\lambda)W_1\sin\beta_1(x/a)\cos n\theta\sin\omega t \qquad (27a)$$

$$q_2 = -\omega^2a\rho_2B(\lambda)W_2\sin[\beta_2(l-x)/a]\cos n\theta\sin\omega t \qquad (27b)$$

$$B(\lambda) = B_b = J_n(\lambda)/\lambda J_n'(\lambda) \qquad (28)$$

Substitution of Eq. (27) into (20) leads finally to the pair of characteristic equations

$$[(1+\mu_iB)\Omega^2 - a_i][\Omega^2 - (n^2+\beta_i^2)] \times \left[\Omega^2 - \frac{1-\nu}{2}(n^2+\beta_i^2)\right] - (n^2+\nu^2\beta_i^2)\Omega^2 + \frac{1-\nu}{2}(n^4 + 2n^2\beta_i^2 + \nu^2\beta_i^4) = 0 \qquad (29)$$

where

$$a_i = 1 + \varkappa(2n^2+\beta_i^2) + \epsilon(n^2+\beta_i^2)^2 \qquad (30)$$

and $\varkappa$, ϵ are defined in Eq. (19).

In the axisymmetric case ($n = 0$), Eqs. (29) and (30) reduce to

$$[(1+\mu_iB)\Omega^2 - a_i](\Omega^2 - \beta_i^2) - \nu^2\beta_i^2 = 0 \qquad (29')$$

$$a_i = 1 + \varkappa\beta_i^2 + \epsilon\beta_i^4 \qquad (30')$$

The solution of the problem requires that Eqs. (23, 28, and 29) be simultaneously satisfied.

Numerical Examples

A. Full Tank

For the axisymmetric case, we write Eq. (18′) in the form

$$\Omega_k^2 = \frac{1}{1+\mu B_{k-1}}\left[a_{33} + \frac{\nu^2\beta^2}{\Omega_{k-1}^2 - \beta^2}\right] \qquad (31a)$$

or

$$\Omega_k^2 = \beta^2\left[1 + \frac{\nu^2}{(1+\mu B_{k-1})\Omega_{k-1}^2 - a_{33}}\right] \qquad (31b)$$

with

$$B_k = B(\lambda_k) = -\frac{J_0(\lambda_k)}{\lambda_kJ_1(\lambda_k)} \qquad (32)$$

$$\lambda_k^2 = \left(\frac{c_s}{c_0}\right)^2\Omega_k^2 - \beta^2 \qquad (33)$$

We shall consider the axisymmetric vibrations of a shell under the following conditions:

1) The shell is filled with a heavy fluid (i.e., a liquid) with

$$\mu = 165 \qquad c_s/c_0 = 6.11$$

2) The shell is filled with a light fluid (i.e., a gas) with

$$\mu = 1.11 \qquad c_s/c_0 = 28.83$$

In addition, for both cases,

$$\nu = 0.3 \qquad \varkappa = 6.3\times10^{-4}$$

$$\epsilon = 5.3\times10^{-7} \qquad \beta = 1.27$$

i.e., the shell, the static pressure, and the longitudinal mode shape are the same.

a) To determine the first natural frequency, we start with $\Omega_0 = 0$ and compute λ_0 and B_0 from Eqs. (33) and (32). We then substitute these values into the right-hand side of Eq. (31a) to compute Ω_1. We repeat these cycles until there is no further change. After a few iteration cycles, we obtain for the first problem: $\Omega = 6.110 \times 10^{-2}$, and the second problem: $\Omega = 4.406 \times 10^{-2}$.

b) To determine the second natural frequency, we start with $\Omega_0 = \beta$ and proceed as before using Eq. (31b) instead of (31a).

Figure 2 shows a general sketch for the graphical solution of the axisymmetric case ($n = 0$). The curves B_a and B_b represent the values of B vs Ω as obtained from the frequency equation (18′) and the definition equation (16). The intersections give the values of Ω, and hence ω, which are the solutions of the system.

B. Partially Filled Tank

Figure 3 shows schematically the curves B_a corresponding to solutions of Eqs. (29′), and the curves B_b representing Eq. (28). The intersections of both sets of curves are the solutions of the system (29′) and (28). In the same figure, the ratios β_1/Ω, β_2/Ω, and λ/Ω are also plotted as functions of Ω.

Using the previously given set of data and assigning the index 1 to the heavier and 2 to the lighter fluid, an iteration procedure similar to the one outlined before yields, for the first intersection, the value

$$\Omega = 6.104 \times 10^{-1}$$

$$\beta_1 = 6.1 \times 10^{-1} \text{ rad}$$

$$\beta_2 = 17.21 \text{ rad}$$

Figure 4 shows a sketch of the solution of the continuity equation (23). The pairs of values l_1, l_2 so obtained determine the different liquid levels for which the system will vibrate at the frequency $\omega^2 = \Omega^2 D/m_s a^2$.

Conclusions

When there is no fluid present, the cylindrical shell has three (two for $n = 0$) natural frequencies associated with a given longitudinal mode shape, whereas for the full tank there can be any number of such natural frequencies. Moreover, the lowest natural frequency of the full tank is always lower than the smaller of the two values $m\pi c_0/l$ and the lowest natural frequency of the shell alone. The highest natural frequency of the full shell is, at most, equal to the highest natural frequency of the shell alone.

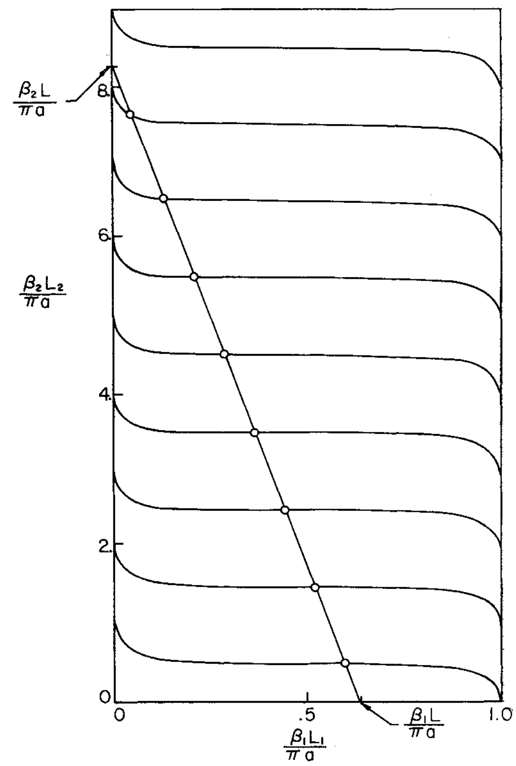

Fig. 4 Partially filled shell: axisymmetric case sketch of solution of continuity equation.

It is of interest to compare the lowest natural frequencies of the full shell with the so called "empty" shell. If we consider the actual case, we see that, in practice, the term "empty" actually refers to a shell filled with air. Thus the so-called empty shell is really a full shell containing a light fluid.

In the numerical examples presented, it can be seen that the lowest natural frequency for the shell filled with a heavy fluid is almost 40% higher than the one for the lighter fluid. This result points to the fact that intuition should not always be trusted in predicting in what case the lowest natural frequency will occur, therefore, a complete analysis of every case should be performed.

When two fluids are present in the shell, we have that, for the axisymmetric case, the frequencies are grouped into a low and a high range. The low range frequencies are lower than the lowest, and the high range frequencies higher than the highest natural frequencies of the corresponding shell without fluid. The case of n other than zero has not been investigated thoroughly, but it is believed that similar conclusions will apply.

For a given circumferential mode shape, there are, in general, several levels of the separation surface associated with the same natural frequency. However, if the liquid level is also prescribed, there may not exist a corresponding natural frequency.

These properties of the partially filled shell solutions are, to say the least, unusual, for they mean that a small change in liquid level is associated with a definite change in the circumferential mode shape (n is an integer) and in the associated frequencies. This change in the circumferential mode shape implies, in turn, that an energy transfer must take place between the shell and the fluids and, given the greater mobility of the latter, instability conditions may be set up which may originate sloshing.

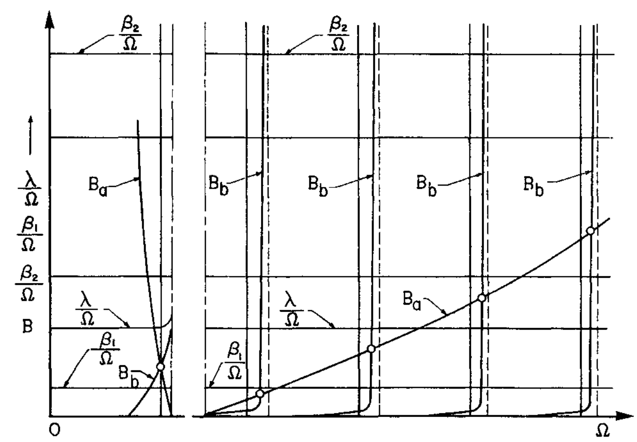

Fig. 3 Partially filled shell: axisymmetric case sketch of solution of characteristic equation.

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